INFLUENCE OF A LONGITUDINALLY COMPRESSED ELASTIC PLATE ON STREAM WAVE PERTURBATION DEVELOPMENT FOR A HOMOGENEOUS FLUID WITH VERTICAL VELOCITY SHIFT

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The influence of an elastic plate on the wave perturbation of a stream with constant velocity over the depth is studied in [1, 2] without taking account of compressive forces, and in [3] in the case of longitudinal tension in the absence of a stream.

1. Let a thin elastic isotropic plate float on the surface of a stream of homogeneous ideal incompressible fluid with a vertical velocity shift. The stream is not perturbed at the initial instant and the plate—fluid surface is horozintal. Starting with the time t = 0, a pressure

$$p = p_0 f(x) \tag{1.1}$$

is applied to the plate surface. We study the process of wave motion development, as generated under longitudinal compression conditions.

Under the assumptions of linear theory with longitudinal compressive forces taken into account [1, 4-6], the problem reduces to solving the system of equations

$$\begin{gathered}\frac{\partial u}{\partial t} + U(z)\frac{\partial u}{\partial x} + \frac{\partial U}{\partial z}w = -\frac{1}{\rho}\frac{\partial p_1}{\partial x},\\ \frac{\partial w}{\partial t} + U(z)\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p_1}{\partial z}, \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\end{gathered} \tag{1.2}$$

with the boundary

$$D_1\frac{\partial^4\zeta}{\partial x^4} + Q_1\frac{\partial^2\zeta}{\partial x^2} + \varkappa_1 F\zeta + \zeta = \frac{1}{\rho g}(p_1 - p) \quad \text{for} \quad z = 0, \tag{1.3}$$

$$w = 0 \quad \text{for} \quad z = -H$$

and initial

$$\zeta = u = w = p_1 = 0 \quad \text{for} \quad t = 0 \tag{1.4}$$

conditions, where

$$F = \frac{\partial^2}{\partial t^2} + 2U\frac{\partial^2}{\partial t\,\partial x} + U^2\frac{\partial^2}{\partial x^2}; \quad D = \frac{Eh^3}{12(1-\mu^2)}; \quad D_1 = \frac{D}{\rho g};$$

$$\varkappa_1 = \frac{\rho_1 h}{\rho g}; \quad Q_1 = \frac{Q}{\rho g};$$

U(z) is the unperturbed stream velocity; ρ, fluid density, g, free-fall acceleration, u, w, horizontal and vertical velocity vector components of the stream wave perturbation; p_1, pressure perturbation; ζ, plate deflection of the rise of the plate-fluid surface; H, tank depth; ρ_1, h, E, μ, density, thickness, normal elastic modulus, and Poisson ratio of the plate; Q, compressive force per unit width of the plate; the x axis is directed vertically upward; and the origin is selected on the unperturbed plate-fluid surface. Here w and ζ are interrelated by the kinematic condition

$$w = \partial\zeta/\partial t + u_0\,\partial\zeta/\partial x, \; u_0 = U(0).$$

We solve problem (1.2)-(1.4) for

$$U(z) = u^* + a(z + H), \tag{1.5}$$

by using the Fourier integral transform in x and the Laplace transform in t. Consequently, after evaluating the Mellin integral and applying the inversion formula, we obtain

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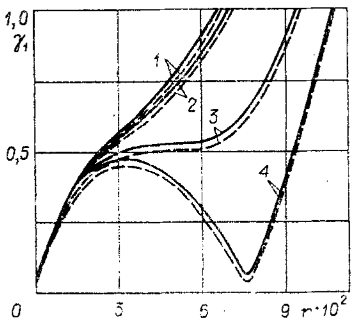

Fig. 1

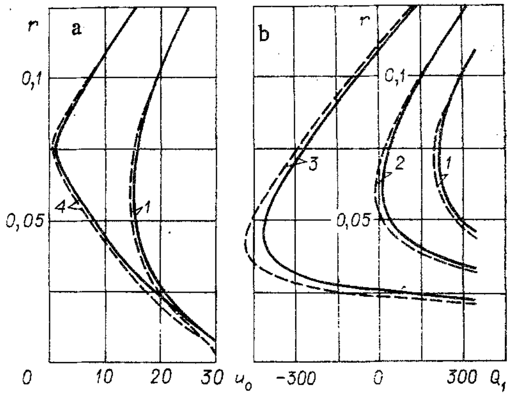

Fig. 2

$$\zeta = \frac{p_0}{\rho\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} l_1^{-1} r f^*(r) \tanh\ rHK(r,t) e^{irx}\, dr,$$

$$K = \frac{1}{\Delta_1\Delta_2} + \frac{1}{(\gamma_1-\gamma_2)\Delta_1} e^{i\Delta_1 t} - \frac{1}{(\gamma_1-\gamma_2)\Delta_2} e^{i\Delta_2 t}, \quad \Delta_{1,2} = \gamma_{1,2} - u_0 r,$$

$$\gamma_{1,2} = \frac{a}{2l_1} \tanh\ rH \pm \tau, \tag{1.6}$$

$$\tau = \left[\left(\frac{1}{4} a^2 \tanh^2 rH + rgl_1 l \tanh\ rH\right) l_1^{-2}\right]^{1/2},$$

$$l_1 = 1 + \varkappa_1 gr \tanh rH, \ l = 1 + D_1 r^4 - Q_1 r^2, \ u_0 = u^* + aH,$$

$f^*(r)$ is the Fourier transform of the function $f(x)$.

Let us evaluate (1.6) under the condition $Q_1 < Q^*$ needed for the stability of an ice floe. Here

$$Q^* = \tau_1(r_0), \quad \tau_1(r) = \frac{a^2 \tanh rH}{4gl_1 r^3} + \frac{1}{r^2} + D_1 r^2,$$

r_0 is the positive root of the equation $\tau'_1(r) = 0$. The quantity Q^* is determined by the vertical velocity shift of the flow, the cylindrical stiffness and density of the plate, the depth and density of the fluid. A large value of Q^* corresponds to a large velocity shift. For values of the parameters

$$E = 3.12\cdot10^9\ \text{N/m}^2,\ \rho_1 = 870\ \text{kg/m}^3, \quad \mu = 0.34, \quad h = 1\ \text{m}, \qquad \rho = 10^3\ \text{kg/m}^3, \quad H = 100\ \text{m}, \tag{1.7}$$

corresponding to an ice floe [1, 7], it equals, in particular, 346.2, 346.4, 346.6 m^2 if a = 0, 0.05, 0.1 m^{-1}. The compressive forces $33.93\cdot10^5$, $33.95\cdot10^5$, $33.97\cdot10^5$ N/m correspond to such values of Q^*. In the absence of a stream and in the case of a stream with constant velocity over the depth $Q^* = 2\sqrt{D_1}$.

Following [8], we rewrite (1.6) in the form

$$\zeta = -\frac{p_0}{\rho\sqrt{2\pi}}(\eta + \eta_1 - \eta_2), \quad \eta = \int\limits_L \frac{r \tanh rH}{\Delta_1\Delta_2} f^*(r) \exp(irx)\, dr,$$

$$\eta_k = \int\limits_L \frac{r \tanh rH}{(\gamma_1-\gamma_2)\Delta_k} f^*(r) \exp(i|x|M_k)\, dr, \quad k = 1, 2, \tag{1.8}$$

$$M_k = (\gamma_k - u_0 r)\nu + r\,\text{sgn}\,x, \quad \nu = t/|x|,$$

where the contours of integration L, L_1, L_2 coincide everywhere with the real axis, except in the neighborhoods of poles of the integrand which are real roots of the equations

$$\gamma_1 - u_0 r = 0, \quad \gamma_2 - u_0 r = 0.$$

The roots of the equation $\gamma_1 - u_0 r = 0$ are bypassed by semicircles on which $\text{Re}(i\Delta_1) > 0$, and the roots of the equation $\gamma_2 - u_0 r = 0$, by semicircles where $\text{Re}(i\Delta_2) > 0$, where L_1 goes around only the point where $\gamma_1 = u_0 r$, and the path L_2 goes around the point where $\gamma_2 = u_0 r$.

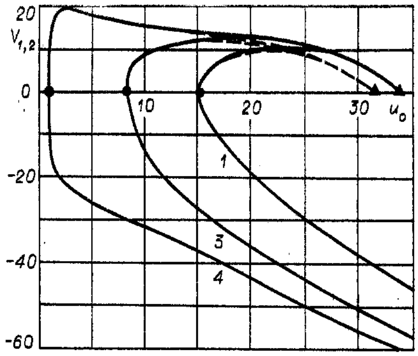

Fig. 3

Under the condition $Q_1 < Q^*$ such a traversal around the poles assures damping of the integrals $\eta_{1,2}$ with time for any fixed x.

Under the considered condition $Q_1 < Q^*$ the equation $\gamma_1 - u_0 r = 0$ has just one real root $r = \alpha_2$ if $u_0 > u_1$, or two roots $r = \alpha_1$, $r = \alpha_2$ for $u_2 < u_0 < u_1$. If $u_0 < u_2$, then the equation mentioned has no real roots. Here

$$u_1 = \frac{1}{2}\left(aH + \sqrt{a^2H^2 + 4gH}\right), \quad u_2 = \gamma_1(r_1)/r_1, \quad u_1 > u_2,$$

and r_1 is the positive root of the equation $r\gamma'_1 - \gamma_1 = 0$.

As regards the equation $\gamma_2 - u_0(r) = 0$, then for $Q_1 < 2\sqrt{D_1}$ by virtue of the equality $\gamma_1(r) = -\gamma_2(-r)$, $\gamma_1(-r) = -\gamma_2(r)$ its roots for corresponding values of u_0 differ from the roots of the equation $\gamma_1 - u_0 r = 0$ only in sign. Under the condition $2\sqrt{D_1} < Q_1 < Q^*$, the equation $\gamma_2 - u_0(r) = 0$ has no other roots except $-\alpha_1$, $-\alpha_2$ despite the fact that $\gamma_2(r)$ has a positive maximum in the domain $r > 0$. In fact $\gamma_2 \leqslant (a/2)\tanh rH$ for $0 < r < \infty$. Moreover, $u_0 \geqslant aH$. Substitution of these estimates reduces the initial equation to the equation $2rH = \tanh rH$, which has no real roots. Therefore, $\gamma_2 \neq u_0(r)$ also for $r > 0$.

2. Let us evaluate integrals (1.8) for the even function f(x) by the method of contour integration with the location of the stationary points (the roots of the equation $M'_{1,2}(r) = 0$) relative to the poles of the integrand taken into account by keeping in mind here that the conditions $\mathrm{Re}(i\Delta_1) \geqslant 0$, $\mathrm{Re}(i\Delta_2) \geqslant 0$ are satisfied on the contour L that bypasses the points $r = \pm\alpha_1$ in the lower and $r = \pm\alpha_2$ in the upper half planes. We consequently obtain the expression

$$\zeta = \begin{cases} O\left(\frac{1}{\sqrt{|x|}}\right), & x < -V_2 t, \\ \zeta_2 + O\left(\frac{1}{\sqrt{|x|}}\right), & -V_2 t < x < 0, \\ \zeta_1 + O\left(\frac{1}{\sqrt{x}}\right), & 0 < x < V_1 t, \\ O\left(\frac{1}{\sqrt{x}}\right), & x > V_1 t \end{cases}$$

under the condition $u_2 < u_0 < u_1$, and the expression

$$\zeta = \begin{cases} \zeta_2 + O\left(\frac{1}{\sqrt{|x|}}\right), & -V_2 t < x < 0, \\ O\left(\frac{1}{\sqrt{|x|}}\right), & x < -V_2 t \end{cases}$$

for $u_0 > u_1$. Here

$$\zeta_k = A_k \sin(\alpha_k |x|), \quad k = 1,2,$$

$$A_k = -\sqrt{2\pi}\,\frac{p_0}{\rho}\,\alpha_k \tau^{-1}(\alpha_k)\, f^*(\alpha_k)\, V_k^{-1} \tanh \alpha_k H, \quad V_1 = u_0 - \gamma_1'(\alpha_1),$$

$$V_2 = \gamma_1'(\alpha_2) - u_0.$$

If $u_0 < u_2$, then $\zeta = O(1/\sqrt{|x|})$ for $x > 0$ and $x < 0$.

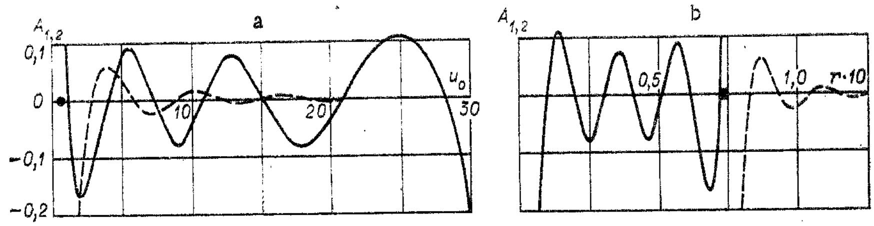

Fig. 4

Therefore, the undamped wave motion of the plate—fluid surface is formed by two waves ζ_1, ζ_2 for $u_2 < u_0 < u_1$, where ζ_2 is elastic (in a fluid with a free surface and in the case of an absolutely flexible plate, it does not occur), while ζ_1 is gravitational. The wave ζ_1 is generated downstream ($x > 0$) and ζ_2 upstream ($x < 0$) of the domain of application of the perturbation (1.1). Under the condition $u_0 > u_1$, the form of the plate-fluid surface is shaped under the effect of just one elastic wave ζ_2 generated in the domain $x < 0$. If $u_0 < u_2$, then the undamped wave motion is not generated either up or downstream. The leading fronts of the waves ζ_1, ζ_2 move from the domain of application of perturbations at the velocities V_1, V_2, respectively, and their lengths are determined by the formulas $\lambda_1 = 2\pi/\alpha_1$, $\lambda_2 = 2\pi/\alpha_2$, where $0 < \lambda_2 < 2\pi/r_1$, $2\pi/r_1 < \lambda_1 < \infty$.

3. Elements of the waves being generated were computed numerically for values of parameters (1.7) and function f(x) equal to one in the domain $|x| \leqslant l$ and zero for $|x| > l$, where $l = 3\cdot 10^2$ m for a quantitative estimation of the influence of longitudinal compression on the vertical velocity shift conditions of a fluid stream.

The dependence of α_1 and α_2 on the magnitude of the compressive force and the vertical velocity shift is illustrated graphically in Fig. 1, where the function $\gamma_1(r)$ is represented in the case $a = 0$ (dashed lines) and $a = 0.05$ m^{-1} (solid lines) for $Q_1 = 0$, 50, 250, 346 m^2 (curves 1-4, respectively). The compressive forces 0, $4.9\cdot 10^5$, $2.45\cdot 10^6$, $3.39\cdot 10^6$ N/m correspond to such Q_1. The wave numbers α_1 and α_2 are abscissas of the intersection of $\gamma_1(r)$ and a line passing through the origin and having the slope u_0. The slope of a line passing through the origin and tangent to $\gamma_1(r)$ from above characterizes u_1, and tangent from below u_2. It is seen that exactly as the growth of the vertical flow velocity gradient, an increase in the compressive force extends the range of values of the velocity u_0, when the undamped waves are generated both up and downstream. Especially significant is the influence of the longitudinal compression. In particular, for $a = 0.05$ m^{-1} the range of $(u_1 - u_2)$ mentioned is 18.9 and 33 m/sec for the compressive forces 0 and $3.39\cdot 10^6$ N/m. If $Q = 3.39\cdot 10^6$ N/m, and the stream velocity does not vary with depth, then $u_1 - u_2 = 30.8$ m/sec, where the change in the difference $u_1 - u_2$ with the change in Q_1 is due mainly to the influence of longitudinal compression on the quantity u_2 which decreases both with the growth of Q_1 and with the diminution of a. For $a = 0.05$ m^{-1} the u_2 equal to 15 and 0.9 m/sec correspond to the values $Q_1 = 0$ and 346 m^2, while $u_2 = 0.54$ m/sec corresponds to the case $Q_1 = 346$ m^2, $a = 0$. The quantity u_1 is practically independent of Q_1. It changes slightly even because of the vertical velocity shift. Indeed the change in a from 0 to 0.05 m^{-1} results in an increase in u_1 by just 8.3% (from 31.3 to 33.9 m/sec).

The influence of the compressive force and the vertical flow velocity shift on the dispersion dependence $r(u_0)$ is illustrated in Fig. 2a, where the notation is the same as in Fig. 1. It follows from the data in Figs. 1 and 2 that the length λ_1 of the gravitational waves ζ_1 occurring only upstream does not exceed the length λ_0 of the elastic waves ζ_2 generated downstream, where max λ_2 = min λ_1. For fixed u_0 and Q_1 taking account of the vertical velocity shift diminishes α_2. However, the influence of elastic waves on the wave number α_2 is slight, and the existing difference in the values of α_2 for $a = 0$ and $a > 0$ vanishes as u_0 grows. The quantity α_1 grows as the vertical flow velocity shift increases. A larger deflection α_1 corresponds to large u_0 for the case $a = 0$ in contrast to $a > 0$. Especially significant are these differences for u_0 close to u_1.

If a is fixed, then a greater α_2 corresponds to large Q_1. The quantity α_1 decreases as Q_1 grows. However, the influence of Q_1 on α_1 diminishes as u_0 increases, and is practically nonexistent for u_0 close to u_1.

The dependence $r(Q_1)$ characterizing the distribution of α_1 and α_2 relative to the magnitude of the compressive force is shown in Fig. 2b for $Q_1 < Q^*$. The solid and dashed lines

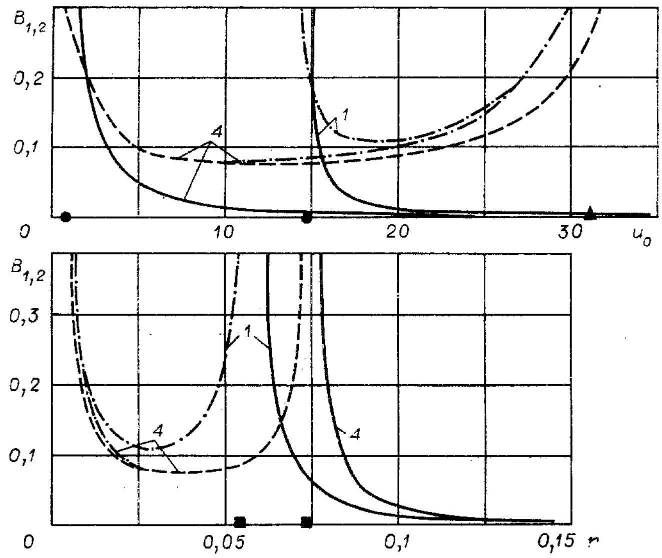

Fig. 5

correspond to the same vertical velocity gradients as in Fig. 2a, and curves 1-3 correspond to u_0 = 10, 15, 21 m/sec. It follows from Fig. 2b that a greater compressive force $Q = Q_2$ corresponds to a smaller u_0, and pressure (1.1) under longitudinal compression conditions generates waves which do not damp out with distance starting with this Q. Indeed, for α = 0.05 m^{-1} the undamped waves are generated for Q greater than 210 and 10 m^2 if u_0 = 10 and 15 m/sec, respectively. In case u_0 = 21 m/sec and α = 0.05 m^{-1}, the undamped waves are generated both in the absence of compressive forces, and under longitudinal compression ($Q_1 > 0$) or tension ($Q_1 < 0$) conditions. An increase in the flow vertical velocity gradient shifts the quantity Q_2 towards lower values. For instance, if waves are generated for u_0 = 15 m/sec only for $Q > 9.8 \cdot 10^4$ N/m in the case α = 0.05 m^{-1}, then in the absence of a stream or a stream with a constant velocity u_0 = 15 m/sec over the depth they also occur for Q = 0 (no compressive forces) and under longitudinal compression ($0 < Q_1 < Q^*$) or tension conditions with the force $0 < Q < 1.96 \cdot 10^5$ N/m. The magnitude of the tension force needed to generate undamped waves by the pressure (1.1) grows as u_0 increases, and decreases with the rise in the flow vertical velocity gradient.

The velocity V_2 of the leading front of the elastic waves is practically independent of the vertical flow profile. It is determined by the elastic forces of the plate and by a compressive force. As regards the frontal velocity V_1 of the gravitational waves, it can then vary even under the effect of longitudinal compression forces and of the vertical flow velocity shift. For $u_0 = u_2$ the equality $V_1 = V_2 = 0$ is satisfied. The velocity V_1 vanishes for $u_0 = u_1$. The frontal velocity V_1 has a maximum in the range $u_2 < u < u_1$ and $V_2 > V_1$. The velocity V_2 of the elastic wave originating for $u_0 > u_2$ grows without limit as u_0 increases. This is shown in Fig. 3, where sections of the curves in the lower half plane, in absolute value, characterize V_2 and in the upper half plane V_1. Values of u_2 are noted by points on the u_0 axis, and values of u_1 by triangles, while the remaining notation is the same as in Fig. 1.

The distribution of the amplitudes A_1 of the gravitational (solid lines) and A_2 of the elastic (dashed lines) waves with respect to u_0 and r is shown in Fig. 4 to the accuracy of the factor $\sqrt{2\pi}p_0$ for α = 0.05 m^{-1}, Q_1 = 346 m^2. Comparing A_1 and A_2 yields a graphic representation of the elastic wave contribution to the wave motion. It is also seen that A_1 and A_2 are oscillating functions of u_0 and r, where the velocity $u_0 = u_2$ (a point on the u_0 axis) is resonant for A_1 and A_2. As u_0 tends to u_1 the amplitude A_1 grows without limit. The value $r = r_1$ (the square on the r axis), to which the wave numbers α_1 and α_2 tend in Fig. 4b, corresponds to the resonant velocity $u_0 = u_2$ in Fig. 4a. If $u_0 \to u_1$, then $\alpha_1 \to 0$. Let us note that changes in r_1 due to the vertical velocity shift do not exceed 3% while an increase in the compressive force results in its substantial growth. In particular, for α = 0.05 m^{-1} the $r_1 \cdot 10^2$ equal to 5.9, 6.1, 7.1, 7.5 m^{-1} correspond to the compressive forces 0, $4.9 \cdot 10^{-5}$, $2.45 \cdot 10^6$, $3.39 \cdot 10^6$ N/m. Let us recall that here u_2 = 15, 14.2, 8.5, 0.9 m/sec, i.e., the change in Q_1 is more substantial on u_2 than on r_1. If α = 0, then u_2 = 14.8, 13.8, 8.2, 0.54 m/sec.

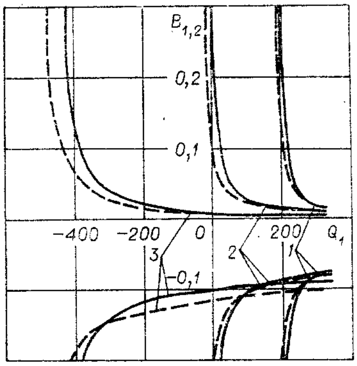

Fig. 6

The contribution of the gravitational and elastic waves to the wave motion is characterized in Fig. 5, where $B_{1,2}(u_0)$ and $B_{1,2}(r)$ are functions governing the greatest possible values of the amplitudes $A_{1,2}(u_0)$ and $A_{1,2}(r)$. The B_2 are superposed by solid, and the B_1 by dashed lines in the graphs for $\alpha = 0.05\ m^{-1}$ and by dash-dot lines for $\alpha = 0$. The corresponding values of u_1, r_1, u_2 for a stream without a velocity shift ($\alpha = 0$) are marked by triangles, squares, and points. Curves 1, 4 correspond to the compressive forces 0 and $3.39 \cdot 10^6$ N/m.

A numerical analysis of the dependence of $B_{1,2}$ on α showed that it does not appear in practice for elastic waves. The exception is the neighborhood of the resonance values, which vary, albeit negligibly, as α changes. The influence of the vertical velocity shift on the quantity B_1 (gravitational waves) is noticeable. It shows up in the diminution of B_1 and grows as u_0 does (diminution of r). This is seen from a comparison of the dashed and dash-dot curve 4 in Fig. 5.

The distribution of $B_{1,2}$ with respect to Q_1 is shown in Fig. 6 for a stream with constant velocity over the depth (dashed lines) and a stream with vertical gradients $\alpha = 0.05$ m^{-1} (solid lines). Curves 1-3 correspond to the same u_0 as in Fig. 2b. It is seen that the possibility of the existence of resonance values of the compressive (or tensile) force is determined by the value of u_0 and the vertical flow velocity shift. As the compressive force increases (starting with resonance) values of $B_{1,2}$ decrease to the critical value resulting in plate buckling. In the case of longitudinal tension, the large amplitude of the elastic and gravitational waves which do not damp with distance corresponds to a large force (before resonance).

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